8 Discrete Random Variables

D0 Intuitively, to tell whether a random variable is discrete, we simply consider the possible values of the random variable. If the random variable is limited to only a finite or countably infinite number of possibilities, then it is discrete.

Example 8.1. Voice Lines: A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used. Let the random variable X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48. [15, Ex 3-1]

D1 **Definition 8.2.** A random variable X is said to be a ***discrete random variable*** if there exists we can find a countable number of distinct real numbers x_k such that

$$\sum_k P[X = x_k] = 1. \tag{11}$$

In other words, X is a discrete random variable if and only if X D2 has a countable support.

Example 8.3. For the random variable N in Example 7.8 (Three Coin Tosses),

The collection of possible values is finite.

The possible values are 0, 1, 2, 3

So, the RV is discrete.

For the random variable S in Example 7.9 (Sum of Two Dice),

The possible values are 2, 3, 4, ..., 12 " "

Example 8.4. Toss a coin until you get a H. Let N be the number of times that you have to toss the coin.

The possible values are 1, 2, 3 ...

The collection of possible values is countably infinite.

So, the RV is discrete.

8.5. Although the support S_X of a random variable X is defined as any set S such that $P[X \in S] = 1$. For discrete random variable, S_X is usually set to be $\{x : P[X = x] > 0\}$, the set of all "possible values" of X.

"Default" support for discrete RV.

Definition 8.6. Important Special Case: An ***integer-valued random variable*** is a discrete random variable whose x_k in (11) above are all integers.

x	$P[X=x]$
x_1	0.25
x_2	0.1
⋮	⋮

8.7. Recall, from 7.20, that the ***probability distribution*** of a random variable X is a description of the probabilities associated with X.

For a discrete random variable, the distribution can be described by just a list of all its possible values $(x_1, x_2, x_3, \ldots)$ along with the probability of each:

$$(P[X = x_1], P[X = x_2], P[X = x_3], \ldots, \text{ respectively}).$$

In many cases, it is convenient to express the probability in the form of a formula. This is especially useful when dealing with a random variable that has infinitely many outcomes. It would be tedious to list all the possible values and the corresponding probabilities.

8.1 PMF: Probability Mass Function

Definition 8.8. When X is a discrete random variable satisfying (11), we define its ***probability mass function*** (pmf) by[32]

$$p_X(x) = P[X = x].$$

$p_X(5) = P[X=5]$ $\qquad$ $p_X(7) = P[X=7]$

- Sometimes, when we only deal with one random variable or when it is clear which random variable the pmf is associated with, we write $p(x)$ or p_x instead of $p_X(x)$.
- The argument (x) of a pmf ranges over *all real numbers*. Hence, the pmf is (and should be) defined for x that is not among the x_k in (11) as well. In such case, the pmf is simply 0. This is usually expressed as "$p_X(x) = 0$, otherwise" when we specify a pmf for a particular random variable.

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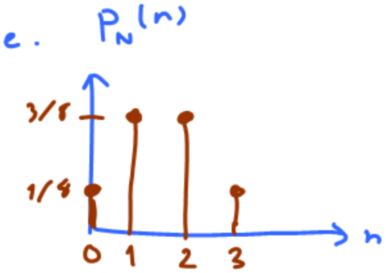
[32] Many references (including [15] and `MATLAB`) does not distinguish the pmf from another function called the probability density function (pdf). These references use the function $f_X(x)$ to represent both pmf and pdf. We will *NOT* use $f_X(x)$ for pmf. Later, we will define $f_X(x)$ as a probability density function which will be used primarily for another type of random variable (continuous RV).

- The pmf of a discrete random variable X is usually referred to as its ***distribution***.

Example 8.9. Continue from Example 7.8. N is the number of heads in a sequence of three coin tosses.

n	$P[N=n]$
0	1/8
1	3/8
2	3/8
3	1/8

$$p_N(n) = \begin{cases} 1/8, & n = 0, 3, \\ 3/8, & n = 1, 2, \\ 0, & \text{otherwise}. \end{cases}$$



8.10. Graphical Description of the Probability Distribution: Traditionally, we use ***stem plot*** to visualize p_X. To do this, we graph a pmf by marking on the horizontal axis each value with nonzero probability and drawing a vertical bar with length proportional to the probability.

8.11. Any pmf $p(\cdot)$ satisfies two properties:

(a) $p(\cdot) \geq 0$

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x.

When you are asked to verify that a function is a pmf, check these two properties.

8.12. Finding probability from pmf: for "any" subset B of $\mathbb{R}$, we can find

$$P[X \in B] = \sum_{x_k \in B} P[X = x_k] = \sum_{x_k \in B} p_X(x_k).$$

In particular, for integer-valued random variables,

$$P[X \in B] = \sum_{k \in B} P[X = k] = \sum_{k \in B} p_X(k).$$

8.13. Steps to find probability of the form P [some condition(s) on X] when the pmf $p_X(x)$ is known.

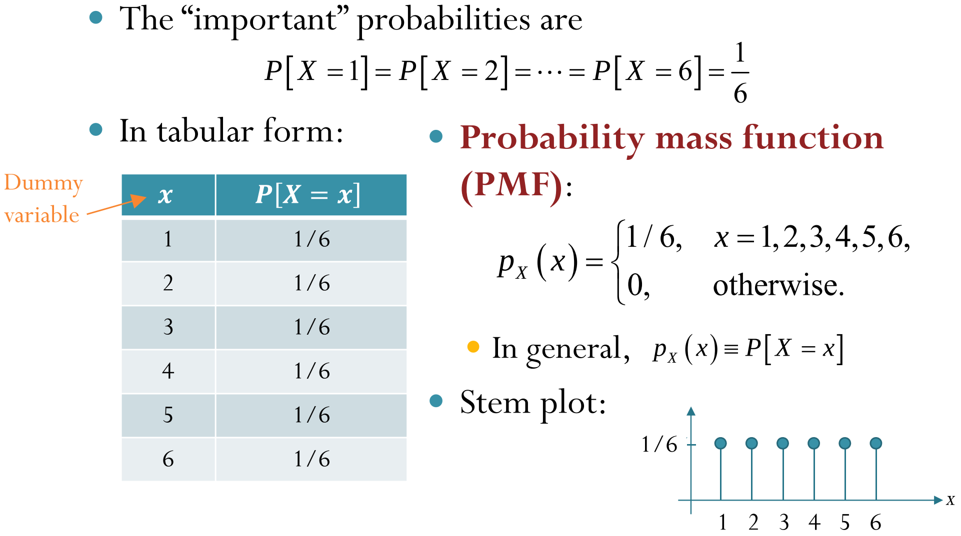
(a) Find the support of X.

(b) Consider only the x inside the support. Find all values of x that satisfy the condition(s).

(c) Evaluate the pmf at x found in the previous step.

(d) Add the pmf values from the previous step.

Example 8.14. Back to Example 7.7 where we roll one dice.



- The "important" probabilities are

$$P[X=1]=P[X=2]=\cdots=P[X=6]=\frac{1}{6}$$

- In tabular form:

Dummy variable

x	$P[X=x]$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

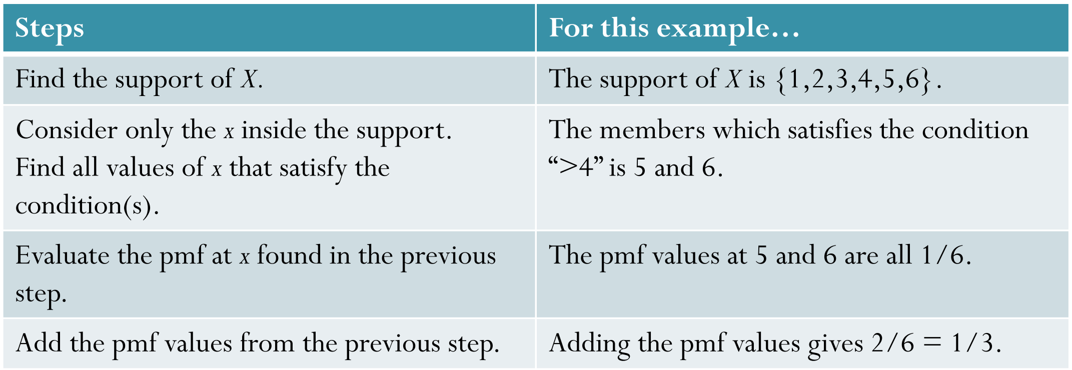
- **Probability mass function (PMF)**:

$$p_X(x)=\begin{cases}1/6, & x=1,2,3,4,5,6,\\ 0, & \text{otherwise.}\end{cases}$$

  - In general, $p_X(x)\equiv P[X=x]$

- Stem plot:

Suppose we want to find $P[X>4]$.



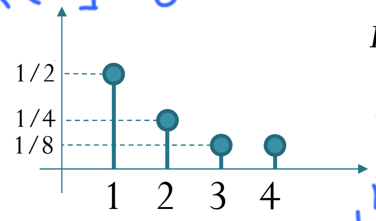
Steps	For this example…
Find the support of X.	The support of X is $\{1,2,3,4,5,6\}$.
Consider only the x inside the support. Find all values of x that satisfy the condition(s).	The members which satisfies the condition ">4" is 5 and 6.
Evaluate the pmf at x found in the previous step.	The pmf values at 5 and 6 are all 1/6.
Add the pmf values from the previous step.	Adding the pmf values gives 2/6 = 1/3.

Example 8.15. Consider a RV X whose $p_X(x) = \begin{cases} 1/2, & x = 1, \\ 1/4, & x = 2, \\ 1/8, & x \in \{3, 4\}, \\ 0, & \text{otherwise.} \end{cases}$

$F_X(3.5) \equiv P[X \leq 3.5] = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$F_X(0) = P[X \leq 0] = 0$

stem plot:



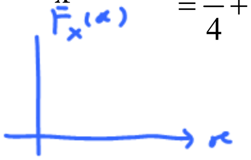
$F_X(1) = \frac{1}{2}$

$F_X(5) = 1$

$$P[X = 2] = p_X(2) = \frac{1}{4}$$

$$P[X > 1] = p_X(2) + p_X(3) + p_X(4)$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$



Example 8.16. Suppose a random variable X has pmf

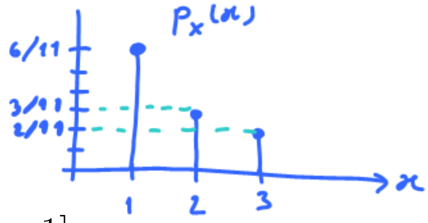
$$p_X(x) = \begin{cases} c/x, & x = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 6/11, & x = 1, \\ 3/11, & x = 2, \\ 2/11, & x = 3, \\ 0, & \text{otherwise} \end{cases}$$

(a) The value of the constant c is

For any pmf, $\left.\sum_x p(x) = 1\right\} \Rightarrow \frac{c}{1} + \frac{c}{2} + \frac{c}{3} + 0 = 1 \Rightarrow c = \frac{1}{1 + \frac{1}{2} + \frac{1}{3}} = \frac{1}{\frac{11}{6}} = \frac{6}{11}$

(b) ✓ Sketch its pmf

carefully



(c) $P[X = 1]$

$= p_X(1) = \frac{c}{1} = \frac{6}{11}$

(d) $P[X \geq 2] = p_X(2) + p_X(3) = \frac{3}{11} + \frac{2}{11} = \frac{5}{11}$

(e) $P[X > 3] = 0$

8.17. Any function $p(\cdot)$ on $\mathbb{R}$ which satisfies

(a) $p(\cdot) \geq 0$, and

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x

is a pmf of some discrete random variable.

8.2 CDF: Cumulative Distribution Function

Definition 8.18. The ***(cumulative) distribution function (cdf)*** of a random variable X is the function $F_X(x)$ defined by

$$F_X(x) = P[X \leq x].$$

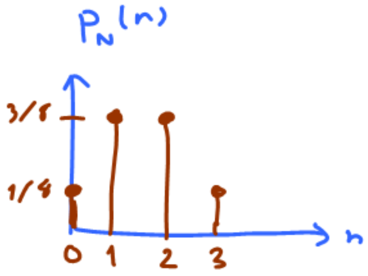
- The argument (x) of a cdf ranges over all real numbers.
- From its definition, we know that $0 \leq F_X \leq 1$.
- Think of it as a function that collects the "probability mass" from $-\infty$ up to the point x.

8.19. From pmf to cdf: In general, for any discrete random variable with possible values $x_1, x_2, \ldots$, the cdf of X is given by

$$F_X(x) = P[X \leq x] = \sum_{x_k \leq x} p_X(x_k).$$

Example 8.20. Continue from Examples 7.8, 7.17, and 8.9 where N is defined as the number of heads in a sequence of three coin tosses. We have

$$p_N(0) = p_N(3) = \frac{1}{8} \text{ and } p_N(1) = p_N(2) = \frac{3}{8}.$$

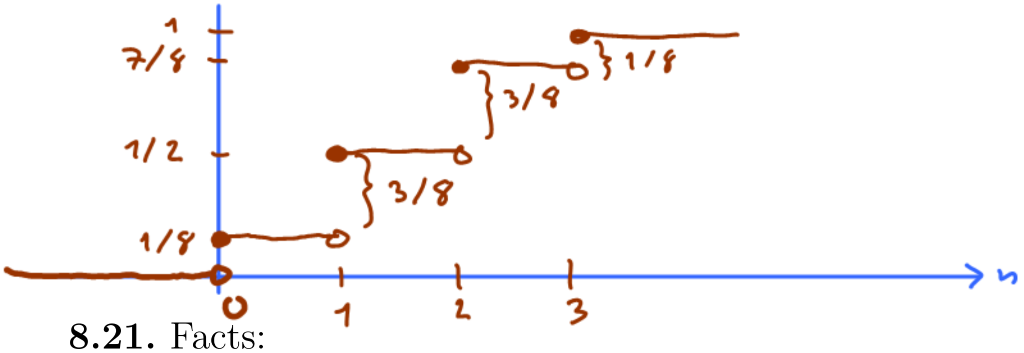


(a) $F_N(0) = P[N \leq 0] = \frac{1}{8}$

(b) $F_N(1.5) = P[N \leq 1.5] = p_N(0) + p_N(1)$

$= \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$

(c) Sketch of cdf $F_N(n) = P[N \le n]$



8.21. Facts:

- For any discrete r.v. X, F_X is a right-continuous, ***staircase*** function of x with jumps at a countable set of points x_k.
- When you are given the cdf of a discrete random variable, you can derive its pmf from the locations and sizes of the jumps. If a jump happens at $x = c$, then $p_X(c)$ is the same as the amount of jump at c. At the location x where there is no jump, $p_X(x) = 0$.

Example 8.22. Consider a discrete random variable X whose cdf $F_X(x)$ is shown in Figure 13.

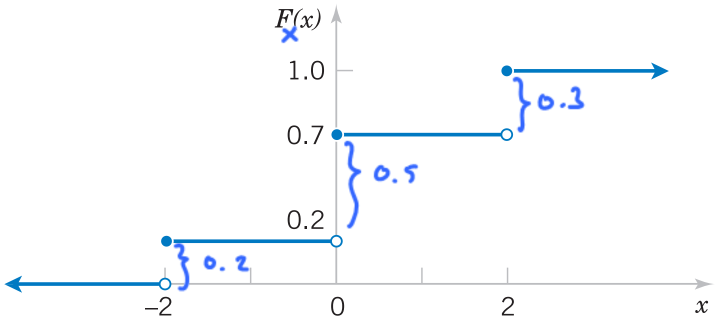


Figure 13: CDF for Example 8.22

Determine the pmf $p_X(x)$. $= \begin{cases} 0.2, & x = -2, \\ 0.5, & x = 0, \\ 0.3, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$

8.23. Characterizing[33] properties of cdf:

CDF1 F_X is non-decreasing (monotone increasing)

$\equiv F_X(x)$ is a nondecreasing function of x

$\equiv$ If $a<b$, then $F_X(a) \leq F_X(b)$

CDF2 F_X is right-continuous (continuous from the right)

$$\equiv \forall x \quad \lim_{y \downarrow x} F_X(y) = F_X(x)$$

$$\equiv \forall x \quad F_X(x^+) = F_X(x)$$

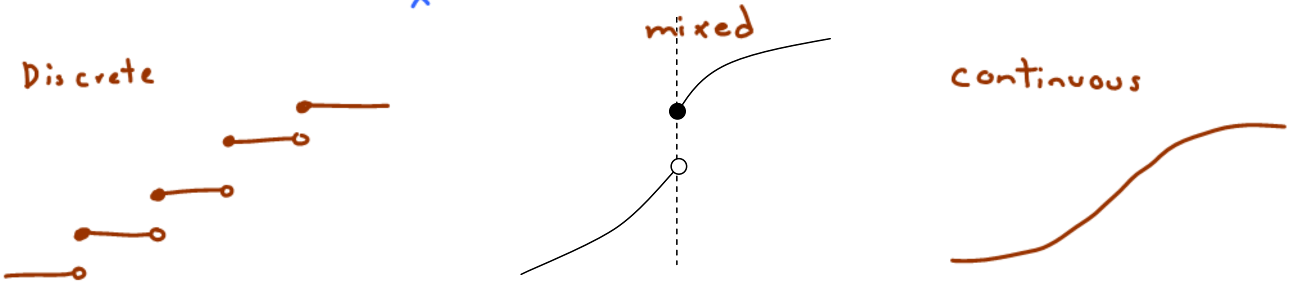


Figure 14: Right-continuous function at jump point

CDF3 $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.

8.24. For discrete random variable, the cdf F_X can be written as

$$F_X(x) = \sum_{x_k} p_X(x_k) u(x - x_k),$$

where $u(x) = 1_{[0,\infty)}(x)$ is the unit step function.

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[33] These properties hold for any type of random variables. Moreover, for any function F that satisfies these three properties, there exists a random variable X whose CDF is F.